The Dirac equation in Cartesian gauge

Ion I. Cotăescu
The West University of Timişoara,
V. Parvan Ave. 4, RO-1900 Timişoara

February 7, 2008

Abstract

It is shown that in the case of the spherically symmetric static backgrounds there is a gauge in which the Dirac equation is manifestly covariant under rotations. This allows us to separate the spherical variables like in the flat space-time, obtaining a pair of radial equations and a specific form of the radial scalar product.

In the gauge field theory [1] on curved space-time the physical meaning does not depend on the choice of the natural (holonomic) frame or on the gauge of the tetrad field which defines the local ones [2, 3]. However, from the technical point of view, these frames are not completely equivalent in the cases when the background has a global symmetry. Then, the symmetry transformations of the tetrad field are generally local, arising from an induced representation [4] of the gauge group for which the group of the global symmetry play the role of a little group. Hence, in such gauges we have to face with the difficulties of the theory of the induced representations which could hide some properties, especially when the form of the field equations depends on the tetrad field, like in the Dirac theory. For this reason it seems that the gauge in which the tetrad field as well as the field equation are manifestly covariant under the global symmetry (i.e. transforming according to linear representations) [2] could offer certain technical advantages.

The spherical symmetric static backgrounds have the global symmetry of the group $T(1) \otimes SO(3)$, of the time translations and the rotations of

the Cartesian space coordinates. There exist a gauge in which the tetrad field in spherical coordinates has only diagonal components and another one where the tetrad field in Cartesian coordinates is manifestly covariant under rotations. This will be referred as the Cartesian gauge. Usually for deriving the Dirac equation one prefers the diagonal tetrad gauge where the result is obtained directly in spherical coordinates [5, 6]. Despite of the obvious advantages of this gauge we believe that the study of the Dirac equation in the Cartesian gauge is also interesting. This will be done here.

We show that in this gauge the Dirac equation can be put in a simple form by using an appropriate transformation of the spinor field [6]. Since this equation is manifestly covariant under rotations we can use the known results from the special relativity in order to separate the spherical variables in terms of the angular momentum eigenspinors [7, 8]. Thus we obtain the radial equations and the form of the radial scalar product in the most general case of a spherically symmetric static metric. We shall work in natural units, $\hbar = c = 1$.

Let us consider a background where we have introduced the natural (holonomic) frame of the coordinates $x^\mu, \mu = 0, 1, 2, 3$. We denote by $e_{\hat\mu}(x)$ the tetrad fields which define the local frames and by $\hat e^{\hat\mu}(x)$ that of the corresponding coframes. These have the usual orthonormalization properties

$$e_{\hat\mu} \cdot e_{\hat\nu} = \eta_{\hat\mu\hat\nu}, \quad \hat e^{\hat\mu} \cdot \hat e^{\hat\nu} = \eta^{\hat\mu\hat\nu}, \quad \hat e^{\hat\mu} \cdot e_{\hat\nu} = \delta^{\hat\mu}_{\hat\nu}, \tag{1}$$

where $\eta =$diag$(1, -1, -1, -1)$ is the Minkowki metric. The 1-forms of the local frames, $d\hat x^{\hat\mu} = \hat e^{\hat\mu}_\nu dx^\nu$, allow one to write the line element

$$ds^2 = \eta_{\hat\mu\hat\nu} d\hat x^{\hat\mu} d\hat x^{\hat\nu} = g_{\mu\nu}(x) dx^\mu dx^\nu. \tag{2}$$

which defines the metric tensor $g_{\mu\nu}$ of the natural frame. This raises or lowers the Greek indices (ranging from 0 to 3) while for the hat Greek ones (with the same range) we have to use the Minkowski metric, $\eta_{\hat\mu\hat\nu}$. The derivatives $\hat\partial_{\hat\nu} = e^\mu_{\hat\nu}\partial_\mu$ satisfy the commutation rules

$$[\hat\partial_{\hat\mu}, \hat\partial_{\hat\nu}] = e^\alpha_{\hat\mu} e^\beta_{\hat\nu} (\hat e^{\hat\sigma}_{\alpha,\beta} - \hat e^{\hat\sigma}_{\beta,\alpha}) \hat\partial_{\hat\sigma} = C_{\hat\mu\hat\nu\cdot}^{\cdot\cdot\hat\sigma} \hat\partial_{\hat\sigma} \tag{3}$$

defining the Cartan coefficients which halp us to write the conecttion components in the local frames as

$$\hat\Gamma^{\hat\sigma}_{\hat\mu\hat\nu} = e^\alpha_{\hat\mu} e^\beta_{\hat\nu} (\hat e^{\hat\sigma}_{\beta,\alpha} + \hat e^{\hat\sigma}_\gamma \Gamma^\gamma_{\alpha\beta}) = \frac{1}{2} \eta^{\hat\sigma\hat\lambda} (C_{\hat\mu\hat\nu\hat\lambda} + C_{\hat\lambda\hat\mu\hat\nu} + C_{\hat\lambda\hat\nu\hat\mu}) \tag{4}$$

while the notation $\Gamma^{\gamma}_{\alpha\beta}$ stands for the usual Christoffel symbols.

Let ψ be a Dirac free field of the mass M, defined on the space domain D. This has the gauge invariant action [9]

$$\mathcal{S}[\psi]=\int_D d^4x\sqrt{-g}\left\{\frac{i}{2}[\bar{\psi}\gamma^{\hat{\alpha}}D_{\hat{\alpha}}\psi-(\overline{D_{\hat{\alpha}}\psi})\gamma^{\hat{\alpha}}\psi]-M\bar{\psi}\psi\right\} \tag{5}$$

where

$$D_{\hat{\alpha}}=\hat{\partial}_{\hat{\alpha}}+\frac{i}{2}S^{\hat{\beta}\cdot}_{\cdot\hat{\gamma}}\hat{\Gamma}^{\hat{\gamma}}_{\hat{\alpha}\hat{\beta}}, \tag{6}$$

are the covariant derivatives of the spinor field and $g=\det(g_{\mu\nu})$. The Dirac matrices, $\gamma^{\hat{\alpha}}$, and the generators of the reducible spinor representation of the $SL(2,C)$ group, $S^{\hat{\alpha}\hat{\beta}}$, satisfy

$$\{\gamma^{\hat{\alpha}},\gamma^{\hat{\beta}}\}=2\eta^{\hat{\alpha}\hat{\beta}}, \qquad [\gamma^{\hat{\alpha}},\gamma^{\hat{\beta}}]=-4iS^{\hat{\alpha}\hat{\beta}}, \tag{7}$$

$$[S^{\hat{\alpha}\hat{\beta}},\gamma^{\hat{\mu}}]=i(\eta^{\hat{\beta}\hat{\mu}}\gamma^{\hat{\alpha}}-\eta^{\hat{\alpha}\hat{\mu}}\gamma^{\hat{\beta}}). \tag{8}$$

Thereby it results that the field equation,

$$i\gamma^{\hat{\alpha}}D_{\hat{\alpha}}\psi-M\psi=0, \tag{9}$$

derived from (5) can be written as

$$i\gamma^{\hat{\alpha}}e^{\mu}_{\hat{\alpha}}\partial_{\mu}\psi-M\psi+\frac{i}{2}\frac{1}{\sqrt{-g}}\partial_{\mu}(\sqrt{-g}e^{\mu}_{\hat{\alpha}})\gamma^{\hat{\alpha}}\psi-\frac{1}{4}\{\gamma^{\hat{\alpha}},S^{\hat{\beta}\cdot}_{\cdot\hat{\gamma}}\}\hat{\Gamma}^{\hat{\gamma}}_{\hat{\alpha}\hat{\beta}}\psi=0. \tag{10}$$

On the other hand, from the conservation of the electric charge we can deduce that when $e^0_i=0$, $i=1,2,3$, then the time-independent relativistic scalar product of two spinors is [9]

$$(\psi,\psi')=\int_D d^3x\mu(x)\bar{\psi}(x)\gamma^0\psi'(x), \quad \mu=\sqrt{-g}e^0_0. \tag{11}$$

Our aim is to discuss here only the case of the spherically symmetric static backgrounds which, as mentioned, have the global symmetry of the $T(1)\otimes SO(3)$ group. These have natural frames of the Cartesian coordinates $x^0=t$ and x^i, $i=1,2,3$, in which the metric tensor is time-independent and manifestly covariant under the rotations $R\in SO(3)$ of the space coordinates,

$$x^{\mu}\to x'^{\mu}=(Rx)^{\nu} \qquad (t'=t, \quad x'^i=R_{ij}x^j). \tag{12}$$

The most general form of a such a metric is given by the line element

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu = A(r)dt^2 - [B(r)\delta_{ij} + C(r)x^i x^j]dx^i dx^j \tag{13}$$

where A, B and C are arbitrary functions of the Euclidian norm of $\vec{x}$, $r = |\vec{x}|$ (which is invariant under rotations). In applications it is convenient to replace these functions by new ones, u, v and w, such that

$$A = w^2, \quad B = \frac{w^2}{v^2}, \quad C = \frac{w^2}{r^2}\left(\frac{1}{u^2} - \frac{1}{v^2}\right). \tag{14}$$

Then the metric appears as the conformal transformation of that simpler one having $w = 1$.

Starting with a Cartesian natural frame we define the Cartesian gauge in which the static tetrad field transforms under the rotations (12) according to the rule

$$d\hat{x}^{\hat{\mu}} \to d\hat{x}'^{\hat{\mu}} = \hat{e}^{\hat{\mu}}_{\alpha}(x')dx'^{\alpha} = (Rd\hat{x})^{\hat{\nu}}. \tag{15}$$

In the case of the metric (13) the simplest choice of their components is

$$\begin{aligned} \hat{e}^0_0 &= \hat{a}(r), \quad \hat{e}^0_i = \hat{e}^i_0 = 0, \quad \hat{e}^i_j = \hat{b}(r)\delta_{ij} + \hat{c}(r)x^i x^j, & (16)\\ e^0_0 &= a(r), \quad e^0_i = e^i_0 = 0, \quad e^i_j = b(r)\delta_{ij} + c(r)x^i x^j, & (17) \end{aligned}$$

where, according to (2), (13) and (14), we must have

$$\begin{aligned} \hat{a} &= w, \quad \hat{b} = \frac{w}{v}, \quad \hat{c} = \frac{1}{r^2}\left(\frac{w}{u} - \frac{w}{v}\right), & (18)\\ a &= \frac{1}{w}, \quad b = \frac{v}{w}, \quad c = \frac{1}{r^2}\left(\frac{u}{w} - \frac{v}{w}\right), & (19) \end{aligned}$$

while the weight function of (11) becomes

$$\mu = \frac{1}{b^2(b + r^2 c)} = \frac{w^3}{uv^2} \tag{20}$$

since

$$\sqrt{-g} = B[A(B + r^2 C)]^{1/2} = \frac{1}{ab^2(b + r^2 c)} = \frac{w^4}{uv^2}. \tag{21}$$

Now we have to replace the concrete form of the tetrad components in Eq.(10). First we eliminate its last term since it is known that this can not

contribute when the metric is spherically symmetric. The argument is that $\{\gamma^{\hat{\alpha}}, S^{\hat{\beta}\hat{\gamma}}\} = \varepsilon^{\hat{\alpha}\hat{\beta}\hat{\gamma}\cdot}_{\cdot\cdot\cdot\hat{\lambda}}\gamma^5\gamma^{\hat{\lambda}}$ (with $\varepsilon^{0123}=1$) is completely antisymmetric while the Cartan coefficients resulted from (16) and (17) have no such components. Furthermore, in order to simplify the remaining equation, we perform the transformation

$$\psi(x) = \chi(r)\hat{\psi}(x). \tag{22}$$

If we chose

$$\chi = [\sqrt{-g}(b + r^2 c)]^{-1/2} = b\sqrt{a} = vw^{-3/2}, \tag{23}$$

then all the terms containing the derivatives of the functions a and b are eliminated (while that of c is not involved). Thus we obtain the Dirac equation in Cartesian gauge,

$$i\{a(r)\gamma^0\partial_t + b(r)(\vec{\gamma}\cdot\vec{\partial}) + c(r)(\vec{\gamma}\cdot\vec{x})[1 + (\vec{x}\cdot\vec{\partial})]\}\hat{\psi}(x) - M\hat{\psi}(x) = 0, \tag{24}$$

expressed only on familiar three-dimensional scalar products and these functions of r. It is clear that this equation is manifestly covariant under rotation and, consequently, all the properties related to the conservation of the angular momentum, including the separation of the variables in spherical coordinates, will be similar as those of the usual Dirac theory in the Minkowski flat space-time.

By using (19) and the traditional notations, $\alpha^i = \gamma^0\gamma^i$ and $\beta = \gamma^0$, we can write the Hamiltonian form of this equation as

$$(\hat{H}\hat{\psi})(x) = i\partial_t\hat{\psi}(x), \tag{25}$$

where the operator $\hat{H}$ with the action

$$\begin{aligned}&(\hat{H}\hat{\psi})(x) = \\ &-i\left\{v(r)(\vec{\alpha}\cdot\vec{\partial}) + \frac{u(r)-v(r)}{r^2}(\vec{\alpha}\cdot\vec{x})[1 + (\vec{x}\cdot\vec{\partial})]\right\}\hat{\psi}(x) + w(r)M\beta\hat{\psi}(x)\end{aligned} \tag{26}$$

is the Hamiltonian of the transformed Dirac field $\hat{\psi}$. Obviously, according to (22), the Hamiltonian of the field ψ is $H = \chi\hat{H}\chi^{-1}$. In our opinion, new interesting particular cases could be anlysed by using this form of the Dirac equation. However, here we restrict ourselves to note only that from (26) we can recover the known result that in the massless limit the Eq.(25) becomes

invariant under the conformal transformations $g_{\mu\nu} \to \hat{w}^2 g_{\mu\nu}$ and $\hat{\psi} \to \hat{w}^{-3/2}\hat{\psi}$ [9] where $\hat{w}$ is an arbitrary function of r.

The last step is to introduce the spherical coordinates, r, θ, ϕ, associated with the space coordinates of our natural Cartesian frame. Doing this we obtain the line element

$$ds^2 = w^2 dt^2 - \frac{w^2}{u^2} dr^2 - \frac{w^2}{v^2} r^2 (d\theta^2 + \sin^2\theta d\phi^2). \tag{27}$$

The form of the Hamiltonian (26) allows us to separate the variables in Eq.(25) as in the case of the central motion in flat space-time, by using the four-components angular spinors $\Phi^{\pm}_{m_j,\kappa_j}(\theta,\phi)$ as given in Ref.[8]. These are orthogonal to each other being completely determined by the quantum number j of the angular momentum, the quantum number m_j of its projection along the third axis and the value of $\kappa_j = \pm(j+1/2)$. Looking for particular (particle-like) solutions of the form

$$\hat{\psi}_{E,j,m_j,\kappa_j}(t,r,\theta,\phi) = \frac{1}{r}[f_+(r)\Phi^{+}_{m_j,\kappa_j}(\theta,\phi) + f_-(r)\Phi^{-}_{m_j,\kappa_j}(\theta,\phi)]e^{-iEt}, \tag{28}$$

after a little calculation, we find the desired radial equations

$$\left[u(r)\frac{d}{dr} + v(r)\frac{\kappa_j}{r}\right] f_+(r) = [E + w(r)M] f_-(r), \tag{29}$$

$$\left[-u(r)\frac{d}{dr} + v(r)\frac{\kappa_j}{r}\right] f_-(r) = [E - w(r)M] f_+(r). \tag{30}$$

Thus, we have shown that in the Cartesian gauge we can separate the radial motion from the angular one which can be completely solved grace to the conservation of the angular momentum. Like in the special relativity, the radial motion here is described by a pair of radial equations which must be solved in each particular case separately. In practice these can be written directly by starting with the metric (27) from which we can identify the functions u, v, and w we need.

The angular spinors are normalized so that the angular integral of the scalar product (11) does not contribute and, consequently, this reduces to a radial integral. By using (22) and (28) we find that this is

$$(\psi, \psi') = \int_{D_r} \frac{dr}{u(r)} [f_+{}^*(r) f'_+(r) + f_-{}^*(r) f'_-(r)] \tag{31}$$

where D_r is the radial domain corresponding to D. What is remarkable here is that the weight function $\mu\chi^2 = 1/u$, resulted from (20) and (23), is just that we need in order to have $(u\partial_r)^+ = -u\partial_r$. Therefore, the operators of the left-hand side of our radial equations are related between them through the Hermitian conjugation with respect to (31). The direct consequence of this property is that the corresponding radial Hamiltonian operator is self-adjoint. Moreover, we observe that if there exists two real constants, c_1 and c_2, so that

$$v = r(c_1 + c_2 w), \tag{32}$$

then this is supersymmetric. Indeed, one can easily verify that a simple rotation in the plane (f_+, f_-) is enough to bring it in the canonical form of a Hamiltonian with supersymmetry [8]. On the other hand, the metrics which satisfy (32) differ from the general ones given by (27) only by a conformal transformation. Thus we can conclude that the central motion of the Dirac particle in the general relativity is up to a conformal transformation a problem with supersymmetry.

References

[1] R. Utiyama, Phys. Rev. **101**, 1597 (1956); T. W. B. Kibble, J. Math. Phys. **2**, 212 (1961)

[2] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, Wiley, New York, 1972

[3] C. M. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation*, W. H. Freeman & Co., San Francisco, 1973

[4] G. Mackey, *Induced Representations of Groups and Quantum Mechanics*, Benjamin, New York, 1968

[5] D. R. Brill and J. A. Wheeler, Rev. Mod. Phys. **29**, 465 (1957); D. R. Brill and J. A. Cohen, J. Math. Phys. **7**, 238 (1966); J. Klauder and J. A. Wheeler, Rev. Mod. Phys. **29**, 516 (1957); T. M. Davis and J. R. Ray, J. Math. Phys. **16**, 75 (1975), Phys. Rev. **D9**, 331 (1974), J. Math. Phys. **16**, 80 (1975); K. D. Kriori and H. Kakati, GRG **20**, 1237 (1995); J. C. Huang, N. O. Santos and Kleber, Class. Quantum Grav. **12**, 1245

(1995); I. D. Soares and J. Tiomno, Phys. Rev. **D54**, 2808 (1996); C. G. De Oliveira and J. Tiomno, Il Nouvo Cimento **24**, 672 (1962); B. D. B. Figueredo, I. D. Soares and Tiomno, Class. Quantum Grav. **9**, 1593 (1992); Hammond R., Class. Quantum Grav. **12**, 279 (1995); P. Baekler, M. Setz, V. Winkelmann, Class. Quantum Grav. **5**, 479 (1988); I. I. Cotăescu and D. Vulcanov, Int. J. Mod. Phys. C **8**, 273 (1997)

[6] V. M. Villalba, preprint gr-qc/9306019

[7] J. D. Bjorken and S. D. Drell S.D. *Relativistic Quantum Mechanics*, McGraw-Hill Book Co., NY, 1964

[8] B. Thaller, *The Dirac Equation*, Springer Verlag, Berlin Heidelberg, 1992

[9] N. D. Birrel and P. C. W. Davies, *Quantum Fields in Curved Space*, Cambridge University Press, Cambridge (1982)